Option Pricing with a Quadratic Diffusion Term

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Discussion Paper No. 226
LSE Financial Markets Group
November 1995

Several authors have derived closed-form option prices in models where the underlying financial variable follows a diffusion process with the following two characteristics: (i) the process has natural upper and lower boundaries; (ii) its diffusion coefficient is quadratic in the current value of the variable. The present paper uses a probabilistic change-of-numeraire technique to compute the corresponding option price formula. In particular, it shows how to interpret the formula in terms of exercise probabilities which are calculated under the martingale measures associated with two specific numeraire portfolios.

Introduction

In the option pricing model of Black and Scholes (1973), the underlying stock price is lognormally distributed, hence has the full positive half-axis as its support. This makes it difficult to apply the Black-Scholes model in situations where the underlying financial variable possesses upper and lower bounds. Ingersoll (1989a, b) for example argues that central bank intervention in the foreign exchange markets will tend to moderate exchange rate fluctuations. He then develops an exchange rate model with strict upper and lower stabilisation bounds, i.e., a model of a perfectly credible target zone regime.

A second area where the assumptions of the Black-Scholes model run into difficulties is the pricing of options on zero-coupon bonds. Indeed, it is well known that modelling bond prices (or bond forward prices) as lognormal variables is tantamount to introducing negative interest rates. This lead Bühler and Käsler (1989) to construct a bond price model within the framework of Merton (1973) where the forward price of the underlying zero-coupon bond is always strictly smaller than 1, so that the corresponding forward interest rate remains positive throughout. More recently, Miltersen,

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Sandmann and Sondermann (1994) have proposed a model of the term structure of interest rates where the forward price of the underlying bond for delivery at the maturity date of the option has risk-neutral dynamics as in the Bühler-Käsler model, while the associated once compounded forward rate follows a lognormal diffusion process.

The structure of the models of Ingersoll (1989a, b), Bühler and Käsler (1989) and Miltersen, Sandmann and Sondermann (1994) is identical in so far as the underlying financial variable is modelled as a diffusion process with the following two characteristics: (i) the process has natural upper and lower boundaries; (ii) its diffusion coefficient is quadratic in the current value of the variable. This specification is easily seen to generalise the Black-Scholes model; in fact, the latter is obtained on choosing zero as lower and $+\infty$ as upper bound.

It is remarkable that this generalisation preserves one of the most attractive features of the Black-Scholes model, namely the existence of analytic formulae for the prices of European call and put options. Ingersoll (1989a, b) and Bühler and Käsler (1989) compute these formulae by applying a judicious change of variable to the corresponding fundamental partial differential equation for pricing derivatives.[1] The present paper, by contrast, applies a probabilistic technique involving a simultaneous change of martingale measure and numeraire which goes back to Jamshidian (1987) and El Karoui and Rochet (1989). This technique makes the different steps in the calculation of the option price more transparent and easier to interpret in financial terms. Moreover, it elucidates the structure of the pricing formula by decomposing the option price in terms of two particular numeraire portfolios and the risk-neutral probabilities associated with these.

The paper is organised as follows. Section 1 sets out the framework of our analysis and introduces the change-of-numeraire technique. Section 2 presents a general expression for the price of a call option in the presence of strict upper and lower bounds on the underlying relative price. Applying this result, Section 3 calculates the call price in models where the underlying relative price has a quadratic diffusion term. Section 4 then shows how the general result applies to the models of Bühler and Käsler (1989), Miltersen, Sandmann and Sondermann (1994) and Ingersoll (1989a, b). Section 5 concludes the paper.

1 Martingale Measures, Numeraires, and Contingent Claims

Fix a finite time interval $\mathcal{T} = [0, T]$, a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ satisfying the usual conditions. $\mathcal{F}_0$ is assumed to be almost trivial, and $\mathcal{F}_T = \mathcal{F}$.

Consider a financial market with continuous and frictionless trade in two primitive assets, labelled 0 and 1, which pay no dividends in $\mathcal{T}$. Let their price processes S^i ($i =$

[1] In fact, there is a slight difference in the approach taken. Ingersoll transforms the fundamental PDE into Merton's (1973) variation of the standard Black-Scholes PDE and then just uses the Black-Scholes solution. Bühler and Käsler transform the fundamental PDE directly into the heat equation and solve the latter in the usual way; see Käsler (1991) or Rady and Sandmann (1994) for details. This is also the approach adopted by Miltersen, Sandmann and Sondermann (1994).

$0, 1)$ be positive semimartingales on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathcal{T}})$. Relative security prices are given by the process $X = S^1/S^0$.

A probability measure Q equivalent to P is called a *martingale measure with respect to asset* 0 if X is a Q-martingale, i.e., if each X_t is Q-integrable and

$$X_t = E^Q [X_T|\mathcal{F}_t]$$

for all $t \in \mathcal{T}$. Alternatively, such a measure Q is said to be *risk-neutral* with respect to asset 0. Let $\mathbb{P}_0$ denote the set of these measures.

Assumption (M) *$\mathbb{P}_0$ is non-empty.*

One element of $\mathbb{P}_0$, denoted Q_0 and called the *reference measure*, will be held fixed throughout the paper.

As in Harrison and Pliska (1983), a vector process $\theta = (\theta^0, \theta^1)$ is called an *admissible trading strategy* if the following properties (i) – (iv) hold:

(i) θ is predictable.

This expresses the informational restriction that trades can only be based on information obtained prior to trading. To formulate the remaining three conditions, let

$$V_t^\theta = \theta_t^0 S_t^0 + \theta_t^1 S_t^1$$

denote the *value process* corresponding to θ.

(ii) V^θ is non-negative.

(iii) θ^1 is integrable with respect to X and the normalised value process satisfies

$$\frac{V_t^\theta}{S_t^0} = \frac{V_0^\theta}{S_0^0} + \int_0^t \theta_s^1 \, dX_s.$$

(iv) The normalised value process V^θ/S^0 is a Q_0-martingale.

Condition (ii) rules out negative portfolio values. Condition (iii) states that all changes in portfolio value are due to the assets' performance rather than to injection or withdrawal of funds. In other words, admissible strategies are self-financing.[2] Condition (iv) says that there are no expected gains from trade. It rules out arbitrage opportunities and certain foolish strategies that throw away money.[3] The space of admissible strategies will be denoted by Θ.

A positive process N is called a *numeraire* if there is a trading strategy $\theta \in \Theta$ such that $N = V^\theta$. Extending our previous definition, we call a probability measure Q

[2]A straightforward integration-by-parts argument shows that (iii) implies the more intuitive representation

$$V_t^\theta = V_0^\theta + \int_0^t \theta_s^0 \, dS_s^0 + \int_0^t \theta_s^1 \, dS_s^1$$

for the value process, provided the integrals exist.

[3]Note that (iv) is the only condition that might depend on the choice of reference measure.

equivalent to P a *martingale measure for numeraire* N (or risk-neutral with respect to N) if V^θ/N, the portfolio value expressed in units of the numeraire, is a Q-martingale for any strategy $\theta \in \Theta$. We shall write $\mathbb{P}_N$ for the set of all such measures, and $\mathbb{P}_1$ if $N = S^1$.

Given the measure Q_0 and a numeraire N, define a probability measure Q_N equivalent to Q_0 (and hence to P) via the Radon-Nikodym derivative

$$\frac{dQ_N}{dQ_0} = \frac{N_T}{N_0}\frac{S_0^0}{S_T^0} \; . \tag{1}$$

Note that N/S^0 is a Q_0-martingale by definition, so the right hand side of (1) has indeed expectation equal to one under Q_0. In case $N = S^1$, we shall write Q_1 for the measure defined by (1).

Lemma 1.1 *Let N be a numeraire and Y a random variable with $\mathrm{E}^{Q_0}[|Y|/S_T^0] < \infty$. Then*

$$\mathrm{E}^{Q_N}\left[\frac{Y}{N_T}\middle|\mathcal{F}_t\right] = \frac{S_t^0}{N_t}\;\mathrm{E}^{Q_0}\left[\frac{Y}{S_T^0}\middle|\mathcal{F}_t\right]$$

for all $t \in \mathcal{T}$.

Proof: The expectation on the left hand side is clearly well-defined and, by a version of the Bayes rule,

$$\mathrm{E}^{Q_N}\left[\frac{Y}{N_T}|\mathcal{F}_t\right] = \frac{\mathrm{E}^{Q_0}\left[\frac{dQ_N}{dQ_0}\frac{Y}{N_T}\middle|\mathcal{F}_t\right]}{\mathrm{E}^{Q_0}\left[\frac{dQ_N}{dQ_0}\middle|\mathcal{F}_t\right]} \; .$$

Using (1) and the fact that $\mathrm{E}^{Q_0}[N_T/S_T^0|\mathcal{F}_t] = N_t/S_t^0$ completes the proof. ■

Applying this lemma to $Y = V_T^\theta$, we see immediately that $Q_N \in \mathbb{P}_N$. We call it the martingale measure obtained from Q_0 by *change of numeraire*. If Q_N and $Q_{\tilde{N}}$ are obtained from Q_0 by changing the numeraire to N and $\tilde{N}$, respectively, then (1) implies

$$\frac{dQ_N}{dQ_{\tilde{N}}} = \frac{N_T}{N_0}\frac{\tilde{N}_0}{\tilde{N}_T} \; . \tag{2}$$

Equations (1) and (2) are at the heart of the change-of-numeraire technique in derivative asset pricing.[4]

A *contingent claim* is a non-negative random variable Γ on $(\Omega, \mathcal{F})$ such that Γ/S_T^0 is Q_0-integrable. A contingent claim is *attainable* if there exists a trading strategy $\theta \in \Theta$ that replicates the claim, i.e., that satisfies $V_T^\theta = \Gamma$. In this case, the portfolio value V_t^θ determines the time t *arbitrage price* $\pi_t(\Gamma)$ of the claim. By property (iv) above, this price can be calculated as

$$\pi_t(\Gamma) = S_t^0\, E^{Q_0}\left[\frac{\Gamma}{S_T^0}\middle|\mathcal{F}_t\right],$$

[4]Cf. El Karoui and Rochet (1989) or Geman, El Karoui and Rochet (1995). For a more detailed examination of the relationship between numeraires and martingale measures see Conze and Viswanathan (1991).

that is, without reference to the replicating strategy. More generally, consider an arbitrary measure $Q \in I\!P_0$ under which Γ / S^0_T is integrable. Independent of whether Γ is attainable or not,

$$\pi^Q_t(\Gamma) = S^0_t \, E^Q\left[\frac{\Gamma}{S^0_T} \middle| \mathcal{F}_t \right]$$

is called the *price under* Q of the claim at time t.[5]

2 European Call Options

Consider an option to receive at time T one unit of asset 1 in exchange for $K > 0$ units of asset 0. This is a slight generalisation of a classical European call option. Indeed, the latter is just the special case where asset 0 is a default-free zero-coupon bond of maturity T.

The option has the following value at the exercise date:[6]

$$\Gamma = \left[S^1_T - K S^0_T \right]^+$$

or, equivalently,

$$\Gamma = (S^1_T - K S^0_T) \, 1_{\mathcal{E}}$$

where

$$\mathcal{E} = \left\{ \omega \in \Omega : \; S^1_T(\omega) > K S^0_T(\omega) \right\}$$

is the event that the option ends 'in the money' and is exercised.

It is well known that the price of a European option can be expressed in terms of exercise probabilities calculated under certain martingale measures. A variant of the following result was derived by El Karoui and Rochet (1989).

Proposition 2.1 *The option price under* Q_0 *is*

$$\pi^{Q_0}_t(\Gamma) = S^1_t \, Q_1(\mathcal{E}|\mathcal{F}_t) - K S^0_t \, Q_0(\mathcal{E}|\mathcal{F}_t)$$

where $Q_1 \in I\!P_1$ *is the measure obtained from* Q_0 *by changing the numeraire to asset 1.*

Proof: By definition,

$$\pi^{Q_0}_t(\Gamma) = S^0_t \, \mathrm{E}^{Q_0}\left[\frac{\Gamma}{S^0_T} \middle| \mathcal{F}_t \right] = S^0_t \, \mathrm{E}^{Q_0}\left[\frac{S^1_T}{S^0_T} 1_{\mathcal{E}} \middle| \mathcal{F}_t \right] - K \, S^0_t \, \mathrm{E}^{Q_0}[1_{\mathcal{E}}|\mathcal{F}_t] \, .$$

Lemma 1.1 implies that

$$S^0_t \, \mathrm{E}^{Q_0}\left[\frac{S^1_T}{S^0_T} 1_{\mathcal{E}} \middle| \mathcal{F}_t \right] = S^1_t \, \mathrm{E}^{Q_1}[1_{\mathcal{E}}|\mathcal{F}_t] \, ,$$

[5]Jacka (1992) shows that a contingent claim Γ is attainable if and only if it has the same initial price $\pi^Q_0(\Gamma)$ under all $Q \in I\!P_0$ for which both dQ_0/dQ and dQ/dQ_0 are bounded. Moreover, he shows that for bounded Γ/S^0_T, the attainability of the claim does not depend on which reference measure Q_0 was used to define the space of admissible trading strategies.

[6]By definition, $[x]^+ = \max\{x, 0\}$ for all real numbers x.

hence the proposition. $\blacksquare$

A different decomposition of the option price can be obtained when the relative price $X = S^1/S^0$ is bounded.

Assumption (B) *There are constants $0 \leq \ell < u \leq +\infty$ such that*

$$\ell S_t^0 < S_t^1 < u S_t^0$$

for all $t \in \mathcal{T}$.

Consider two portfolios, the first of which is long one unit of asset 0 and short u^{-1} units of asset 1, while the second is long one unit of asset 1 and short ℓ units of asset 0.[7] Let

$$U = S^0 - u^{-1} S^1$$

and

$$L = S^1 - \ell S^0$$

denote the corresponding value processes. Under Assumption (B), these are positive processes, hence numeraires.

Proposition 2.2 *Under Assumption* (B)*, the option price under Q_0 is*

$$\pi_t^{Q_0}(\Gamma) = \frac{1}{1 - u^{-1}\ell} \left\{ (1 - u^{-1}K)\, L_t\, Q_L(\mathcal{E}|\mathcal{F}_t) - (K - \ell)\, U_t\, Q_U(\mathcal{E}|\mathcal{F}_t) \right\}$$

where $Q_U \in I\!P_U$ and $Q_L \in I\!P_L$ are the measures obtained from Q_0 by changing the numeraire to U and L, respectively.

Proof: It is straightforward to check that

$$S_T^1 - K S_T^0 = \frac{(1 - u^{-1}K)\, L_T - (K - \ell)\, U_T}{1 - u^{-1}\ell} \,.$$

Thus,

$$\pi_t^{Q_0}(\Gamma) = \frac{1 - u^{-1}K}{1 - u^{-1}\ell}\, S_t^0\, \mathrm{E}^{Q_0}\left[\frac{L_T}{S_T^0}\, 1_{\mathcal{E}} \middle| \mathcal{F}_t \right] \; - \; \frac{K - \ell}{1 - u^{-1}\ell}\, S_t^0\, \mathrm{E}^{Q_0}\left[\frac{U_T}{S_T^0}\, 1_{\mathcal{E}} \middle| \mathcal{F}_t \right].$$

Lemma 1.1 now implies

$$S_t^0\, \mathrm{E}^{Q_0}\left[\frac{L_T}{S_T^0}\, 1_{\mathcal{E}} \middle| \mathcal{F}_t \right] = L_t\, \mathrm{E}^{Q_L}[1_{\mathcal{E}}|\mathcal{F}_t]$$

and

$$S_t^0\, \mathrm{E}^{Q_0}\left[\frac{U_T}{S_T^0}\, 1_{\mathcal{E}} \middle| \mathcal{F}_t \right] = U_t\, \mathrm{E}^{Q_U}[1_{\mathcal{E}}|\mathcal{F}_t] \,.$$

This is the desired result. $\blacksquare$

[7] Of course, u^{-1} is understood to be zero if $u = +\infty$.

We have again expressed the call price as a function of certain exercise probabilities, this time evaluated under martingale measures associated with the numeraires U and L.

The exercise event can be characterised in terms of the random variable $Y_T = L_T/U_T$ or its inverse $Z_T = U_T/L_T$:

$$\begin{aligned}
\mathcal{E} &= \left\{\omega \in \Omega : \ Y_T(\omega) > \frac{K-\ell}{1-u^{-1}K}\right\} \\
&= \left\{\omega \in \Omega : \ Z_T(\omega) < \frac{1-u^{-1}K}{K-\ell}\right\}.
\end{aligned}$$

Ingersoll (1989a, b), Bühler and Käsler (1989) and Miltersen, Sandmann and Sondermann (1994) propose models where the law of the processes $Y = L/U$ and $Z = U/L$ under Q_U and Q_L is very simple, so that the above exercise probabilities are easy to determine.

3 Models with a Quadratic Diffusion Coefficient

The following assumption postulates that after a change of measure, relative asset prices follow a diffusion process with quadratic diffusion coefficient. We shall see later that the models mentioned at the end of the previous section are of this type. Let constants $\sigma > 0$ and $0 \le \ell < u \le +\infty$ be given.

Assumption (Q) *There exists a Q_0-Wiener process W^0 such that the process of relative asset prices $X = S^1/S^0$ solves the stochastic differential equation*

$$dX_t = \sigma\,(X_t - \ell)(1 - u^{-1}X_t)\,dW_t^0$$

with initial value $\ell < X_0 < u$.

Standard results from the theory of stochastic processes imply that the above stochastic differential equation has in fact a solution. This solution is unique both in the strong and weak sense, satisfies Assumption (B) and is a martingale; see for example Revuz and Yor (1991) and Karlin and Taylor (1981). In particular, Q_0 is indeed risk-neutral with respect to asset 0.

Note that the lognormal dynamics of Black and Scholes (1973) and Merton (1973) are obtained as the special case where $\ell = 0$ and $u = +\infty$.

3.1 Characterisation

It turns out that Assumption (Q) can be formulated equivalently in terms of the processes $Y = L/U$ or $Z = U/L$. Let $Q_U \in \mathbb{P}_U$ and $Q_L \in \mathbb{P}_L$ be the measures obtained from Q_0 by changing the numeraire to U and L, respectively, and define $\hat{\sigma} = (1-u^{-1}\ell)\sigma$.

Lemma 3.1 *Assumption (Q) is equivalent to each of the following two properties:*

(i) *There exists a Q_U-Wiener process W^U such that Y solves*

$$dY_t = \hat{\sigma}\, Y_t\, dW_t^U$$

with initial value $Y_0 > 0$.

(ii) *There exists a Q_L-Wiener process W^L such that Z solves*

$$dZ_t = \hat{\sigma}\, Z_t\, dW_t^L$$

with initial value $Z_0 > 0$.

PROOF: Suppose Assumption (Q) holds. By Itô's lemma and some algebra,[8]

$$dY_t = \hat{\sigma} Y_t \left\{ dW_t^0 + \tilde{\sigma}\,(X_t - \ell)\, dt \right\}$$

where $\tilde{\sigma} = u^{-1}\sigma$. Define a process W^U by $dW_t^U = dW_t^0 + \tilde{\sigma}\,(X_t - \ell)\, dt$ with $W_0^U = 0$. We want to show that W^U is a Wiener process under Q_U. By equation (2),

$$\frac{dQ_U}{dQ_0} = \frac{U_T}{U_0}\frac{S_0^0}{S_T^0} = \frac{1 - u^{-1}X_T}{1 - u^{-1}X_0} \,.$$

On the other hand,

$$\frac{d[1 - u^{-1}X_t]}{1 - u^{-1}X_t} = -\tilde{\sigma}\,(X_t - \ell)\, dW_t^0,$$

hence, by the formula for the martingale exponential,

$$1 - u^{-1}X_t = (1 - u^{-1}X_0)\, \exp\left(-\tilde{\sigma} \int_0^t (X_s - \ell)\, dW_s^0 - \frac{\tilde{\sigma}^2}{2}\int_0^t (X_s - \ell)^2\, ds \right).$$

In particular,

$$\frac{dQ_U}{dQ_0} = \exp\left(-\tilde{\sigma} \int_0^T (X_s - \ell)\, dW_s^0 - \frac{\tilde{\sigma}^2}{2}\int_0^T (X_s - \ell)^2\, ds \right).$$

The Girsanov theorem now implies that W^U is indeed a Q_U-Wiener process; cf. Revuz and Yor (1991).

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[8] The following facts are used in the calculations. If

$$y = \frac{x - \ell}{1 - u^{-1}x} \,,$$

then

$$\frac{dy}{dx} = \frac{1 - u^{-1}\ell}{(1 - u^{-1}x)^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{2u^{-1}(1 - u^{-1}\ell)}{(1 - u^{-1}x)^3} \,.$$

Moreover,

$$\frac{dx}{dy} = \frac{(1 - u^{-1}x)^2}{1 - u^{-1}\ell} \quad \text{and} \quad \frac{d^2x}{dy^2} = \frac{-2u^{-1}(1 - u^{-1}x)^3}{(1 - u^{-1}\ell)^2} \,.$$

To prove the converse implication (i) $\Rightarrow$ (Q), suppose we have W^U as in the lemma. Itô's lemma and some straightforward computations yield

$$dX_t = \sigma\,(X_t - \ell)(1 - u^{-1}X_t)\left\{dW_t^U - \tilde{\sigma}\,(X_t - \ell)\,dt\right\}.$$

Let W^0 be the process defined by $dW_t^0 = dW_t^U - \tilde{\sigma}\,(X_t - \ell)\,dt$ with $W_0^0 = 0$. As

$$\frac{d[1 - u^{-1}X_t]}{1 - u^{-1}X_t} = -\tilde{\sigma}\,(X_t - \ell)\,dW_t^U + \tilde{\sigma}^2\,(X_t - \ell)^2\,dt,$$

the formula for the martingale exponential now implies

$$1 - u^{-1}X_t = (1 - u^{-1}X_0)\,\exp\left(-\tilde{\sigma}\int_0^t (X_s - \ell)\,dW_s^U + \frac{\tilde{\sigma}^2}{2}\int_0^t (X_s - \ell)^2\,ds\right)$$

and

$$\frac{dQ_0}{dQ_U} = \frac{1 - u^{-1}X_0}{1 - u^{-1}X_T} = \exp\left(\tilde{\sigma}\int_0^T (X_s - \ell)\,dW_s^U - \frac{\tilde{\sigma}^2}{2}\int_0^T (X_s - \ell)^2\,ds\right).$$

By the Girsanov theorem, W^0 is a Wiener process under Q_0.

Next, we want to show that (i) implies (ii). Let W^U be a Q_U-Wiener process as in the statement of the lemma. By the formula for the martingale exponential,

$$Y_T = Y_0\,\exp\left(\hat{\sigma}\,W_T^U - \frac{1}{2}\hat{\sigma}^2 T\right).$$

Define a process W^L by $dW_t^L = -dW_t^U + \hat{\sigma}\,dt$ with $W_0^L = 0$. As

$$\frac{dQ_L}{dQ_U} = \frac{U_0}{L_0}\frac{L_T}{U_T} = \frac{Y_T}{Y_0} = \exp\left(\hat{\sigma}\,W_T^U - \frac{1}{2}\hat{\sigma}^2 T\right),$$

the Girsanov theorem implies that W^L is a Wiener process under Q_L. By construction,

$$Y_T = Y_0\,\exp\left(-\hat{\sigma}\,W_T^L + \frac{1}{2}\hat{\sigma}^2 T\right),$$

hence

$$Z_T = Z_0\,\exp\left(\hat{\sigma}\,W_T^L - \frac{1}{2}\hat{\sigma}^2 T\right).$$

In other words, $dZ_t = \hat{\sigma}\,Z_t\,dW_t^L$.

The converse implication (ii) $\Rightarrow$ (i) follows in the same way. ■

Thus, Assumption (Q) holds if and only if there is a change of measure that makes the process Y (or Z) a driftless geometric Brownian motion whose 'volatility' (i.e., instantaneous standard deviation of returns) is $\hat{\sigma}$. This is the key to our calculation of the option price.

3.2 The Option Price

Let $(\mathcal{G}_t)_{t\in\mathcal{T}}$ be the filtration generated by the process X, and set $\mathcal{G}=\mathcal{G}_T$. The following result is well known.

Proposition 3.1 *Under Assumption* (Q), *any contingent claim* Γ *with* $\mathcal{G}$*-measurable normalised payoff* Γ/S_T^0 *is attainable.*

PROOF: This is an immmediate consequence of the martingale representation property of X on $(\Omega,\mathcal{G},Q_0,(\mathcal{G}_t)_{t\in\mathcal{T}})$; see Revuz and Yor (1991). ■

This guarantees in particular attainability of the option to receive one unit of asset 1 in exchange for K units of asset 0, as its normalised payoff $[S_T^1-KS_T^0]^+/S_T^0=[X_T-K]^+$ is clearly measurable with respect to $\mathcal{G}$.[9] Let Φ denote the standard normal distribution function.

Proposition 3.2 *Under Assumption* (Q), *the option to receive one unit of asset* 1 *in exchange for* K *units of asset* 0 *is attainable. For* $\ell<K<u$, *its time* t *arbitrage price is*

$$\pi_t(\Gamma)=\frac{1}{1-u^{-1}\ell}\left\{(1-u^{-1}K)\,(S_t^1-\ell S_t^0)\,\Phi(e_t^+)-(K-\ell)\,(S_t^0-u^{-1}S_t^1)\,\Phi(e_t^-)\right\}$$

where

$$e_t^{\pm}=\frac{1}{\hat\sigma\sqrt{T-t}}\left[\log\frac{S_t^1-\ell S_t^0}{S_t^0-u^{-1}S_t^1}-\log\frac{K-\ell}{1-u^{-1}K}\pm\frac{1}{2}\hat\sigma^2\,(T-t)\right]$$

and $\hat\sigma=(1-u^{-1}\ell)\sigma$.

PROOF: We want to apply Proposition 2.2, so let Q_U and Q_L be the measures obtained from Q_0 by changing the numeraire to U and L, respectively. To calculate the probability of exercise under Q_U and Q_L, let W^U and W^L be Wiener processes as in Lemma 3.1, so that

$$Y_t=Y_0\,\exp\left(\hat\sigma\,W_t^U-\frac{1}{2}\hat\sigma^2 t\right)$$

and

$$Z_t=Z_0\,\exp\left(\hat\sigma\,W_t^L-\frac{1}{2}\hat\sigma^2 t\right)$$

by the formula for the martingale exponential.

The properties of the Wiener process W^U now imply

$$\begin{aligned}
Q_U(\mathcal{E}|\mathcal{F}_t) &= Q_U\left(Y_T>\frac{K-\ell}{1-u^{-1}K}\,\middle|\,Y_t\right)\\
&= Q_U\left(\log Y_T-\log Y_t>\log\frac{K-\ell}{1-u^{-1}K}-\log Y_t\right)
\end{aligned}$$

[9] For $u<+\infty$, moreover, the normalised payoff of the option is bounded, so attainability does not depend on which reference measure was chosen to define the space of admissible trading strategies; cf. Jacka (1992). To see this in the case $u=+\infty$, consider a put option and use put-call parity.

$$
\begin{aligned}
&= Q_U\Big(\hat\sigma\,(W_T^U - W_t^U) > \log\frac{K-\ell}{1-u^{-1}K} - \log Y_t + \frac{1}{2}\hat\sigma^2\,(T-t)\Big) \\
&= \Phi\left(\frac{1}{\hat\sigma\sqrt{T-t}}\left[\log Y_t - \log\frac{K-\ell}{1-u^{-1}K} - \frac{1}{2}\hat\sigma^2\,(T-t)\right]\right).
\end{aligned}
$$

In the same way, we find

$$
\begin{aligned}
Q_L(\mathcal{E}|\mathcal{F}_t) &= Q_L\Big(Z_T < \frac{1-u^{-1}K}{K-\ell}\Big|\, Z_t\Big) \\
&= Q_L\Big(\hat\sigma\,(W_T^L - W_t^L) < \log\frac{1-u^{-1}K}{K-\ell} - \log Z_t + \frac{1}{2}\hat\sigma^2\,(T-t)\Big) \\
&= \Phi\left(\frac{1}{\hat\sigma\sqrt{T-t}}\left[\log Y_t - \log\frac{K-\ell}{1-u^{-1}K} + \frac{1}{2}\hat\sigma^2\,(T-t)\right]\right).
\end{aligned}
$$

This completes the proof. ■

Standard arguments[10] show that the trading strategy

$$
\begin{aligned}
\theta_t^0 &= \frac{1}{1-u^{-1}\ell}\Big\{-(1-u^{-1}K)\,\ell\,\Phi(e_t^+) - (K-\ell)\,\Phi(e_t^-)\Big\} \\
\theta_t^1 &= \frac{1}{1-u^{-1}\ell}\Big\{(1-u^{-1}K)\,\Phi(e_t^+) + (K-\ell)\,u^{-1}\,\Phi(e_t^-)\Big\}
\end{aligned}
$$

is admissible and replicates the option.

For $\ell = 0$ and $u = +\infty$, we obtain of course the option price formula of Black and Scholes (1973) and Merton (1973) with $\hat\sigma = \sigma$. Setting $u = +\infty$ but $\ell > 0$ leads to a formula proposed by Rubinstein (1983).

The result is easily extended to allow a time-dependent, but deterministic, parameter function $\sigma(t) > 0$ in Assumption (Q). Lemma 3.1 then holds with $\hat\sigma$ replaced by $\hat\sigma(t) = (1-u^{-1}\ell)\,\sigma(t)$, and the term $\hat\sigma\sqrt{T-t}$ in Proposition 3.2 must be replaced with

$$
(1-u^{-1}\ell)\,\sqrt{\int_t^T \sigma^2(s)\,ds}.
$$

The price of a generalised put option, that is, an option to *give up* one unit of asset 1 in exchange for K units of asset 0, can be calculated in the same way. Alternatively, one can use a version of put-call parity.

4 Examples

This section shows how the models of Bühler and Käsler (1989), Miltersen, Sandmann and Sondermann (1994) and Ingersoll (1989a, b) fit into the framework developed in the previous sections.

[10] See for instance Harrison and Pliska (1981).

4.1 Options on Zero-Coupon Bonds

Fix dates $T' > T > 0$ and let assets 0 and 1 be pure discount bonds without default risk, maturing at T and T', respectively. Without loss of generality, their face values can be normalised to 1, i.e., $S^0_T = 1$ and $S^1_{T'} = 1$. Consider a standard European call option written on bond 1 with exercise price K and exercise date T. As $S^0_T = 1$, this call can be considered as an option to receive one unit of bond 1 in exchange for K units of bond 0.

Bühler and Käsler (1989) propose a model where the bond prices satisfy $S^0_t < 1$ for $t < T$ and $S^1_t < S^0_t$ for $t \leq T$. These inequalities follow directly from the postulate that interest rates implied by bond prices ought to be positive. In fact, the former inequality means that the interest rate for a loan from t to T is positive, while the latter states that the forward interest rate, as seen at time t, for the period from T to T' is positive. In particular, Assumption (B) holds with $u = 1$ and $\ell = 0$.

More specifically, the relative price $X_t = S^1_t / S^0_t$ has the form

$$X_t = \left[1 + \frac{1 - h(t)}{h(t)}\, e^{-\sigma W_t}\right]^{-1}$$

where $h : \mathcal{T} \to]0, 1[$ is a continuously differentiable function, σ a positive constant and W a standard Wiener process under the measure P. The process S^0 is defined similarly, but need not be specified here. Note that X_t is the time t *forward price* of bond 1 for delivery at time T. It is easily seen that $h(t)$ is the median value of this forward price.

We want to show that this model satisfies Assumption (Q). Itô's lemma yields

$$dX_t = \sigma\, X_t\, (1 - X_t)\ \{\alpha_t\, dt + dW_t\}$$

with the bounded process

$$\alpha_t = \frac{h'(t)}{\sigma h(t)[1 - h(t)]} + \sigma\, \left(\frac{1}{2} - X_t\right).$$

Define a process W^0 by

$$dW^0_t = \alpha_t\, dt + dW_t$$

and $W^0_0 = 0$, and let Q_0 be the measure obtained via the Radon-Nikodym derivative

$$\frac{dQ_0}{dP} = \exp\left(-\int_0^T \alpha_s\, dW_s - \frac{1}{2}\int_0^T \alpha_s^2\, ds\right)$$

(as α is a bounded process, the random variable on the right hand side has indeed expectation equal to 1). The Girsanov theorem implies that W^0 is a Wiener process under Q_0. By construction, $dX_t = \sigma\, X_t\, (1 - X_t)\, dW^0_t$, so Assumption (Q) holds.

By Proposition 3.2, the arbitrage price of the call option with exercise price $0 < K < 1$ is

$$\pi_t(\Gamma) = (1 - K)\, S^1_t\, \Phi(e^+_t) - K\, (S^0_t - S^1_t)\, \Phi(e^-_t)$$

with

$$e_t^{\pm} = \frac{1}{\sigma\sqrt{T-t}} \left[\log \frac{S_t^1}{S_t^0 - S_t^1} - \log \frac{K}{1-K} \pm \frac{1}{2}\sigma^2 (T-t) \right].$$

This is the pricing formula derived by Bühler and Käsler (1989).

Miltersen, Sandmann and Sondermann (1994) obtain the same option price formula in a model of the term structure of interest rates. To see how their approach fits into the framework studied in the present paper, note that the variable

$$Z_t = \frac{1 - X_t}{X_t} = X_t^{-1} - 1$$

can be interpreted as the *once compounded forward rate*, as seen at time t, for a loan given at T and repaid at T'. Miltersen, Sandmann and Sondermann start from lognormal diffusion dynamics for the forward rate Z:

$$dZ_t = \mu(t) Z_t \, dt + \sigma(t) Z_t \, dW_t$$

with deterministic functions μ and $\sigma > 0$, and a Wiener process W under some measure P. This can be rewritten as

$$dZ_t = \sigma(t) Z_t \, dW_t^L$$

where W^L is the process defined by

$$dW_t^L = dW_t + \frac{\mu(t)}{\sigma(t)} \, dt$$

with $W_0^L = 0$. Granted sufficient regularity of the parameter functions,[11] the Girsanov theorem implies that W^L is a Wiener process under the measure Q_L obtained via the Radon-Nikodym derivative

$$\frac{dQ_L}{dP} = \exp\left(- \int_0^T \frac{\mu(s)}{\sigma(s)} \, dW_s - \frac{1}{2} \int_0^T \frac{\mu^2(s)}{\sigma^2(s)} \, ds \right).$$

According to our earlier results, this implies Assumption (Q) with the time-dependent parameter function $\sigma(t)$, hence the time-dependent volatility version of the Bühler-Käsler bond option formula.

4.2 Currency Options in a Target Zone Regime

Consider an option to buy at some future date T one unit of a foreign currency for K units of the domestic currency. If asset 0 is a default-free domestic discount bond paying one domestic currency unit at time T, and asset 1 its foreign counterpart, then the currency option can be interpreted as the right to receive one unit of asset 1 in exchange for K units of asset 0. Note that S^1, the *domestic* price of asset 1, is the product of two factors: the spot exchange rate s, giving the number of domestic currency units needed to purchase one unit of the foreign currency, and $S^{1,f}$, the price

[11] Boundedness of the ratio μ/σ will do.

of asset 1 in *foreign* units. Assuming for simplicity that the domestic interest rate r_d and the foreign interest rate r_f are constant, we clearly have

$$S_t^0 = e^{-r_d\,(T-t)}, \quad S_t^{1,f} = e^{-r_f\,(T-t)} \quad \text{and} \quad S_T^1 = s_t\, e^{-r_f\,(T-t)}.$$

By covered interest rate parity, $X_t = S_t^1/S_t^0$ is now just the time t forward rate for currency exchange at time T.

Ingersoll (1989a) models a perfectly credible target zone regime by imposing the condition

$$\xi(t) < s_t < \Xi(t)$$

with deterministic functions ξ and Ξ. He shows that not every pair of boundary functions is admissible. Given $\xi(0)$ and $\Xi(0)$, the tightest possible bounds are in fact

$$\begin{aligned} \xi(t) &= \xi(0)\, e^{(r_d - r_f)t}, \\ \Xi(t) &= \Xi(0)\, e^{(r_d - r_f)t}. \end{aligned}$$

For these functions, the above condition translates into

$$\xi(T) S_t^0 < S_t^1 < \Xi(T) S_t^0,$$

that is, Assumption (B) with $\ell = \xi(T)$ and $u = \Xi(T)$.

As for the spot rate dynamics, one of the models studied in Ingersoll (1989a) has

$$ds_t = \mu_t s_t\, dt + \sigma\, [s_t - \xi(t)][1 - s_t/\Xi(t)]\, dW_t$$

with an unspecified drift rate process μ, a Wiener process W and the above boundary functions. By Itô's lemma, the corresponding forward rate dynamics are

$$dX_t = (\mu_t + r_f - r_d) X_t\, dt + \sigma\, [X_t - \xi(T)][1 - X_t/\Xi(T)]\, dW_t,$$

which, under suitable conditions on μ, implies Assumption (Q). If so, the arbitrage price of the currency option is given by Proposition 3.2 and can be written as

$$\begin{aligned} \pi_t(\Gamma) &= [s_t - \xi(t)]\, S_t^{1,f}\, \frac{1 - K/\Xi(T)}{1 - \xi(T)/\Xi(T)}\, \Phi(e_t^+) \\ &\quad - [1 - s_t/\Xi(t)]\, S_t^0\, \frac{K - \xi(T)}{1 - \xi(T)/\Xi(T)}\, \Phi(e_t^-) \end{aligned}$$

with

$$e_t^{\pm} = \frac{1}{\hat{\sigma}\sqrt{T-t}} \left[\log \frac{s_t - \xi(t)}{1 - s_t/\Xi(t)} - \log \frac{K - \xi(T)}{1 - K/\Xi(T)} \pm \frac{1}{2} \hat{\sigma}^2\, (T-t) \right]$$

and $\hat{\sigma} = [1 - \xi(T)/\Xi(T)]\sigma$. This is the same result as in Ingersoll (1989a).

An extension of this analysis to 'futures-style' options (futures contracts on option payoffs) is presented in Ingersoll (1989b). Assuming a quadratic diffusion term for the underlying futures price, Ingersoll calculates valuation formulae similar to the one above. Again, the results of Sections 2 and 3 apply.

5 Conclusion

We have studied the pricing of a European-type option to exchange one asset for another in the presence of strict upper and lower bounds on the relative price of these assets. Our first result shows how to decompose the option price in terms of two particular numeraire portfolios and the probabilities of exercise under the martingale measures associated with these numeraires. This decomposition is particularly useful in models where the relative asset price has a quadratic diffusion coefficient. The second contribution of the paper is a new derivation of the option price in this class of models.

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